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8. The functions f and g are defined by

$$\mathrm{f}: x \mapsto 2x + \ln 2, \quad x \in \mathbb{R}, \qquad \mathrm{g}: x \mapsto \mathrm{e}^{2x}, \quad x \in \mathbb{R}.$$

(*a*) Prove that the composite function gf is

$$\mathrm{gf}: x \mapsto 4\mathrm{e}^{4x}, \quad x \in \mathbb{R}.$$ **(4)**

(*b*) Sketch the curve with equation $y = \mathrm{gf}(x)$, and show the coordinates of the point where the curve cuts the y-axis.
(1)

(*c*) Write down the range of gf.
(1)

(*d*) Find the value of x for which $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{gf}(x)] = 3$, giving your answer to 3 significant figures. **(4)**

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5. (*a*) Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, prove that

$$\cos 2A \equiv 1 - 2\sin^2 A.$$ **(2)**

(*b*) Show that

$$2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 \equiv \sin\theta(4\cos\theta + 6\sin\theta - 3).$$ **(4)**

(*c*) Express $4\cos\theta + 6\sin\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. **(4)**

(*d*) Hence, for $0 \leq \theta < \pi$, solve

$$2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1),$$

giving your answers in radians to 3 significant figures, where appropriate. **(5)**

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7. The points A and B have position vectors $\mathbf{i} - \mathbf{j} + p\mathbf{k}$ and $7\mathbf{i} + q\mathbf{j} + 6\mathbf{k}$ respectively, where p and q are constants.

The line l_1, passing through the points A and B, has equation

$$\mathbf{r} = 9\mathbf{i} + 7\mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \text{ where } \lambda \text{ is a parameter.}$$

(*a*) Find the value of p and the value of q. **(4)**

(*b*) Find a unit vector in the direction of $\overrightarrow{AB}$. **(2)**

A second line l_2 has vector equation

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}), \text{ where } \mu \text{ is a parameter.}$$

(*c*) Find the cosine of the acute angle between l_1 and l_2. **(3)**

(*d*) Find the coordinates of the point where the two lines meet. **(5)**

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4. $$\mathrm{f}(x) = (x^2 + 1)\ln x, \qquad x > 0.$$

(*a*) Use differentiation to find the value of $\mathrm{f}'(x)$ at $x = \mathrm{e}$, leaving your answer in terms of e.

(4)

(*b*) Find the exact value of $\int_1^{\mathrm{e}} \mathrm{f}(x)\ \mathrm{d}x$.

(5)

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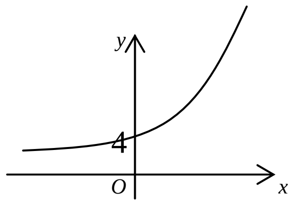
(*a*)	$\text{gf}(x) = e^{2(2x+\ln 2)}$		M1	
	$= e^{4x}e^{2\ln 2}$		M1	
	$= e^{4x}e^{\ln 4}$		M1	
	$= 4e^{4x}$		A1 cso	(4)
(*b*)		Shape and point	B1	(1)
(*c*)	Range is $\mathbb{R}^+$	Accept $\text{gf}(x) > 0,\ y > 0$	B1	(1)
(*d*)	$\frac{d}{dx}[\text{gf}(x)] = 16e^{4x}$			
	$e^{4x} = \frac{3}{16}$		M1 A1	
	$4x = \ln\frac{3}{16}$		M1	
	$x \approx -0.418$		A1	(4)
				Total 10

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(*a*) $\cos 2A = \cos^2 A - \sin^2 A$ (+ use of $\cos^2 A + \sin^2 A \equiv 1$)	M1	
$= (1 - \sin^2 A); - \sin^2 A = 1 - 2\sin^2 A$ (*)	A1	(2)
(*b*) $2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3$		
$\equiv 4\sin\theta\cos\theta; -3(1 - 2\sin^2\theta) - 3\sin\theta + 3$	B1; M1	
$\equiv 4\sin\theta\cos\theta + 6\sin^2\theta - 3\sin\theta$	M1	
$\equiv \sin\theta(4\cos\theta + 6\sin\theta - 3)$ (*)	A1	(4)
(*c*) $4\cos\theta + 6\sin\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$		
Complete method for *R* (may be implied by correct answer)		
[$R^2 = 4^2 + 6^2$, $R\sin\alpha = 4$, $R\cos\alpha = 6$]	M1	
$R = \sqrt{52}$ or 7.21	A1	
Complete method for α; $\alpha = 0.588$ (allow 33.7º)	M1 A1	(4)

P.T.O. for (*d*)

(*d*) $\sin\theta\ (4\cos\theta + 6\sin\theta - 3) = 0$ M1

$\theta = 0$ A1

$\sin(\theta + 0.588) = \dfrac{3}{\sqrt{52}} = 0.4160..$ (24.6º) M1

$\theta + 0.588 = (0.4291),\ 2.7125$ [or $\theta + 33.7° = (24.6º),\ 155.4°$] dM1

$\theta = 2.12$ A1 (5)

Total 15

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(*a*) Solves $9+2\lambda=1$ *or* $7+2\lambda=-1$ to give $\lambda=-4$ so $p = 3$ M1 A1

Solves $9+2\lambda=7$ *or* $7+\lambda=6$ to give $\lambda=-1$ so $q = 5$ M1 A1 (4)

(*b*) $|6\text{i}+6\text{j}+3\text{k}|=9$ so unit vector is $\frac{1}{9}(6\mathbf{i}+6\mathbf{j}+3\mathbf{k})$ M1 A1 (2)

(*c*) $\cos\theta=\dfrac{2\times2+2\times1+1\times2}{3\times3}$ M1 A1

$\therefore \cos\theta=\frac{8}{9}$ A1 (3)

(*d*) Write down two of $9+2\lambda=3+2\mu,\ 7+2\lambda=2+\mu$ or $7+\lambda=3-2\mu$

B1 B1

Solve to obtain $\mu=1$ or $\lambda=-2$ M1 A1

Obtain coordinates (5, 3, 5) A1 (5)

Total 14

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(*a*) $\text{f}'(x)=(x^2+1)\times\frac{1}{x}+\ln x\times 2x$ M1 A1

$\text{f}'(\text{e})=(\text{e}^2+1)\times\frac{1}{\text{e}}+2\text{e}=3\text{e}+\frac{1}{\text{e}}$ M1 A1 (4)

(*b*) $(\dfrac{x^3}{3}+x)\ln x-\int(\dfrac{x^3}{3}+x)\dfrac{1}{x}dx$ M1 A1

$= (\dfrac{x^3}{3}+x)\ln x-\int(\dfrac{x^2}{3}+1)dx$

$= \left[(\dfrac{x^3}{3}+x)\ln x-(\dfrac{x^3}{9}+x)\right]_1^e$ A1

$= \frac{2}{9}e^3+\frac{10}{9}$ M1 A1 (5)

Total 9